ALLOWANCE FOR GAS BLOWING IN SUPERSONIC FLOW OVER A WEDGE

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UDC 533.601.15

The static pressure on the dividing streamline in intense blowing of a gas through the wall of a wedge in a supersonic flow is determined.

In several investigations ([1,2], and others) the experimental data were presented basically in the form of angle of inclination of the contact surface. Bott [1] discussed the effect of the viscosity on the dividing streamline on the experimental results, but presented no numerical results for the influence of this factor on the shape of the dividing streamline.

The present author attempts to show how to determine the numerical effect of viscosity during intense blowing on a wedge on the position of the contact surface, using the relations at the shock wave and a correlation of [1].

We consider the exact relation of [3] for a perfect gas; this relates the flow deviation and the pressure difference in passing through a shock wave. When gas is blown through the wall of the wedge and the shock wave is attached, one can write this relation in the form

$$\operatorname{tg}(\theta+\Delta\theta)=\sqrt{M_\infty^2-1}\,\frac{C_p}{2-C_p}\left[\frac{1-\frac{\gamma+1}{2}\cdot\frac{M_\infty^2}{M_\infty^2-1}\cdot\frac{C_p}{2}}{1+\frac{\gamma+1}{2}M_\infty^2\frac{C_p}{2}}\right]^{\frac{1}{2}}. \tag{1}$$

Carrying out a transformation in Eq. (1), we obtain

$$f(\bar{p})=C_p^3-\frac{4}{\gamma+1}\left[1+\gamma\sin^2(\theta+\Delta\theta)-\frac{1}{M_\infty^2}\right]C_p^2+$$
$$+4\left(1-\frac{4}{\gamma+1}\,\frac{1}{M_\infty^2}\right)\sin^2(\theta+\Delta\theta)\,C_p+\frac{16\sin^2(\theta+\Delta\theta)}{(\gamma+1)M_\infty^2}=0, \tag{2}$$

where $\Delta\theta$, according to [1], is given by the relation

$$\operatorname{tg}\Delta\theta=\frac{m}{Bp}\left(\frac{R_0}{M}T_w\right)^{\frac{1}{2}}.$$

Since the pressure can be determined by calculating the viscous interaction, we can solve Eq. (2) by the method of successive approximations.

1. First Approximation ($Re_x=\infty$)

Even in this simplest case it is difficult to solve Eq. (2) in the general form. Therefore, we use a result from thin body hypersonic theory, which gives only an approximation, but a value so close to the true root $\bar{p}$, that we can use certain methods to obtain further improvement. Let

$$\Delta\theta\approx\frac{\psi}{\bar{p}},\quad \psi=\frac{m\left(\frac{R_0}{M}T_w\right)^{\frac{1}{2}}}{Bp_\infty}.$$

Kiev Institute of Civil Aviation Engineers. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 30, No. 6, pp. 1017-1024, June, 1976. Original article submitted January 13, 1975.

Then

$$\bar{p} = 1 + \gamma M_\infty^2 \left(\theta + \frac{\psi}{\bar{p}}\right)^2 \left[\frac{\gamma+1}{4} + \sqrt{\left(\frac{\gamma+1}{4}\right)^2 + \frac{1}{M_\infty^2\left(\theta+\frac{\psi}{\bar{p}}\right)^2}}\,\right]. \tag{3}$$

Since $\bar{p} \geq 1$, the quantity $\frac{\gamma-1}{\gamma+1}$ can be neglected in comparison with $\bar{p}$ and we obtain the following result from Eq. (3):

$$\bar{p}^3 - \left[\frac{4+\gamma(\gamma+1)K_\theta^2}{2}\right]\bar{p}^2 + [1-\gamma(\gamma+1)K_\theta K_\psi]\,\bar{p} - \frac{\gamma(\gamma+1)}{2}K_\psi^2 = 0. \tag{4}$$

We solve this equation by the method described in [4].

Introducing the notation

$$x = \alpha\bar{p},\ a = -\alpha\,\frac{4+\gamma(\gamma+1)K_\theta^2}{2},\quad b = \alpha^2[1-\gamma(\gamma+1)K_\theta K_\psi],\ \alpha = \sqrt[3]{\frac{2}{\gamma(\gamma+1)K_\psi^2}}$$

into Eq. (4), we obtain

$$f(x) = x^3 + ax^2 + bx - 1 = 0. \tag{5}$$

If $\alpha > 2$, then

$$\frac{1}{2\sqrt{\gamma(\gamma+1)}} > K_\psi \geqslant 0,\ -\alpha K_\psi + \sqrt{\alpha^2 K_\psi^2 + \frac{2\alpha-4}{\gamma(\gamma+1)}} > K_\theta \geqslant 0 \tag{6}$$

and the function $f(x)$ changes sign in the interval [0, 1]. Here $f(0) < 0$, $f(1) > 0$. From the last inequalities and the expression $x_1 x_2 x_3 = 1$ it follows that only one real root falls between 0 and 1. Using the third Chebyshev polynomial

$$T_3(x) = 32x^3 - 48x^2 + 18x - 1, \tag{7}$$

constructed in the interval [0, 1], we can reduce Eq. (5) to a quadratic. The root of this equation lying in the range [0, 1] gives a value for the dimensionless pressure $\bar{p} = \bar{p}_*$ less than 1. Therefore, we find the desired root $\bar{p}$ from the equation

$$\bar{p}^2 + \left(\bar{p}_* + \frac{a}{\alpha}\right)\bar{p} + \bar{p}_*\left(\bar{p}_* + \frac{a}{\alpha}\right) + \frac{b}{\alpha^2} = 0.$$

Since of the two roots of this equation one is physically nonreal (less than 1), we finally obtain

$$\bar{p}_1 = \sqrt{\left(1 + \frac{\gamma(\gamma+1)K_\theta^2}{4} - \frac{\bar{p}_*}{2}\right)^2 + \gamma(\gamma+1)K_\theta K_\psi - 1 + \bar{p}_*\left(2 + \frac{\gamma(\gamma+1)K_\theta^2}{2} - \bar{p}_*\right)} + 1 + \frac{\gamma(\gamma+1)K_\theta^2}{4} - \frac{\bar{p}_*}{2}, \tag{8}$$

where

$$\bar{p}_* = \frac{(0.5625 - b) + \sqrt{(0.5625-b)^2 + 4\cdot 0.9687\left(a + \frac{3}{2}\right)}}{2\alpha\left(a + \frac{3}{2}\right)}.$$

If $\alpha < 2$, then

$$K_\psi > \frac{1}{2\sqrt{\gamma(\gamma+1)}},\ K_\theta \geqslant 0 \tag{9}$$

and the function $f(x)$ will still be negative at the point $x=1$, and so the root of Eq. (5) will be located in the range $[1, \infty]$. We map this interval using the transformation $\bar{x} = \frac{1}{x}$ in the range [0, 1]. In this case Eq. (5) becomes the equation

$$f(\bar{x}) = -\bar{x}^3 + b\bar{x}^2 + a\bar{x} + 1 = 0. \tag{10}$$

Simultaneous consideration of Eqs. (7) and (10) allows us to choose a root located in the range [0, 1] and to use it and the original equation (4) to find one physically possible solution for the pressure coefficient in the form

$$\bar{p}_1 = \frac{2\alpha^{-1}\left(\frac{3}{2} - b\right)}{a + 0{,}5625 + \sqrt{(a + 0{,}5625)^2 + 4\cdot 0{,}9687\left(\frac{3}{2} - b\right)}} . \tag{11}$$

The approximate values of the root $\bar{p}_1$ of Eq. (2), determined from Eqs. (8) and (11), can be improved by the method of [4], and one finally obtains

$$\bar{p} = \bar{p}_1 + h_1, \tag{12}$$

where

$$\frac{1}{h_1} = -\frac{f'(\bar{p}_1)}{f(\bar{p}_1)} + \frac{1}{2}\frac{f''(\bar{p}_1)}{f'(\bar{p}_1)} .$$

The function $f(\bar{p})$ is chosen using Eq. (2) at the point $\bar{p} = \bar{p}_1$. The derivatives of this function for $\bar{p} = \bar{p}_1$ have the form

$$f'(\bar{p}_1) = \frac{2C_{p_1}}{\gamma M_\infty^2}\left[3C_{p_1} - \frac{8}{\gamma+1}\left(1 - \frac{1}{M_\infty^2}\right)\right] - \frac{8\sin^2(\theta + \Delta\theta)}{M_\infty^2} \times$$
$$\times\left(1 - \frac{4}{\gamma+1}\,\frac{1}{M_\infty^2}\right)\left(\frac{2C_{p_1}}{\gamma+1} - \frac{1}{\gamma}\right) + \frac{4\psi\sin 2(\theta + \Delta\theta)}{\bar{p}_1^2 + \psi^2} \times$$
$$\times\left[\frac{\gamma C_{p_1}^2}{\gamma+1} - C_{p_1}\left(1 - \frac{4}{\gamma+1}\,\frac{1}{M_\infty^2}\right) - \frac{4}{(\gamma+1)M_\infty^2}\right],$$

$$f''(\bar{p}_1) = \frac{8}{\gamma^2 M_\infty^4}\left[3C_{p_1} - \frac{4}{\gamma+1}\left(1 - \frac{1}{M_\infty^2}\right) - \right.$$
$$\left. - \frac{4\gamma\sin^2(\theta + \Delta\theta)}{\gamma+1}\left(1 - \frac{4}{\gamma+1}\,\frac{1}{M_\infty^2}\right)\right] - \frac{8\psi\cos 2(\theta + \Delta\theta)}{\bar{p}_1^2 + \psi^2} \times$$
$$\times\left\{\frac{\psi + \bar{p}_1\,\mathrm{tg}\,2(\theta + \Delta\theta)}{\bar{p}_1^2 + \psi^2}\left[\frac{\gamma C_{p_1}^2}{\gamma+1} - C_{p_1}\left(1 - \frac{4}{\gamma+1}\,\frac{1}{M_\infty^2}\right) - \frac{4}{(\gamma+1)M_\infty^2}\right] - \frac{2\,\mathrm{tg}\,2(\theta + \Delta\theta)}{M_\infty^2} \times\right.$$
$$\left.\times\left(\frac{2C_{p_1}}{\gamma+1} - \frac{8C_{p_1}}{(\gamma+1)^2 M_\infty^4} + \frac{4}{\gamma(\gamma+1)M_\infty^4} - \frac{1}{\gamma}\right)\right\},$$

where

$$\Delta\theta = \mathrm{arctg}\,\frac{\psi}{\bar{p}_1}, \quad C_{p_1} = \frac{2(\bar{p}_1 - 1)}{\gamma M_\infty^2} .$$

The results, determined using Eq. (12) for all $\theta + \Delta\theta$, allowed in shock theory, and $M_\infty > 2.5$, give a highly accurate root $\bar{p}$ of Eq. (2), since the values of $f(\bar{p}_1 + h_1)$ in our calculations for $\theta + \Delta\theta$ from 5 to 40° and M_∞ from 2.5 to 20 did not exceed unity in absolute magnitude in the fifth decimal place. Thus, the first approximation corresponds to $\bar{p}_1 + h_1$ and $\theta + \Delta\theta_1$, where

$$\Delta\theta_1 = \mathrm{arctg}\,\frac{\psi}{\bar{p}_1 + h_1} .$$

2. Second Approximation (Weak Interaction)

Applying the arguments of [5] to our case, we note that weak interactions will appear for small $\theta + \Delta\theta_1$ and large Mach and Reynolds numbers, or for moderate supersonic Mach numbers and low Reynolds numbers. These interactions will also appear for large Mach numbers and large values of $\theta + \Delta\theta_1$. As was true in [5], the flow without viscous effects will be denoted by the subscript "first," apart from the local parameters $K_\psi = M_\infty\psi$ and $K_{\theta\psi} = M_\infty(\theta + \Delta\theta)$. We shall take the pressure to be constant across the boundary layer and write it without a subscript. Since weak interactions are characterized by the perturbation of initial flow conditions caused by the hypersonic boundary layer, the ratio of local pressure on the dividing streamline to the local pressure in the inviscid flow can be represented as a power series in $\frac{d\delta^*}{dx}$ in the form

$$\frac{p}{p_{\text{first}}} = 1 + \frac{p_\infty}{p_{\text{first}}}\left(\frac{d\frac{p}{p_\infty}}{d\theta}\right)_{\theta=\theta_\psi}\frac{d\delta^*}{dx} + \frac{1}{2}\frac{p_\infty}{p_{\text{first}}}\left(\frac{d^2\frac{p}{p_\infty}}{d\theta^2}\right)_{\theta=\theta_\psi}\left(\frac{d\delta^*}{dx}\right)^2 + \cdots, \tag{13}$$

where $\frac{p_{\text{first}}}{p_\infty} = \bar{p}_1 + h_1$ and is given by Eq. (12).

The first coefficients of the series (13) in the tangent wedge approximation without correction for centrifugal forces can be calculated from Eq. (2) and have the form

$$\left(\frac{d\frac{p}{p_\infty}}{d\theta}\right)_{\theta=\theta_\psi} = \frac{\frac{2\gamma M_\infty^2 \sin 2\theta_\psi}{\gamma+1}\left[\left(\gamma+1-\frac{4}{M_\infty^2}\right)C_p - \gamma C_p^2 + \frac{4}{M_\infty^2}\right]}{\frac{8C_p}{\gamma+1}\left(1+\sin^2\theta_\psi - \frac{1}{M_\infty^2}\right) - 3C_p^2 - 4\left(1-\frac{4}{\gamma+1}\cdot\frac{1}{M_\infty^2}\right)\sin^2\theta_\psi}, \tag{14}$$

$$\left(\frac{d^2\frac{p}{p_\infty}}{d\theta^2}\right)_{\theta=\theta_\psi} = 2\left(\frac{d\frac{p}{p_\infty}}{d\theta}\right)_{\theta=\theta_\psi} \times$$

$$\times\left\{\operatorname{ctg}2\theta_\psi + \frac{\frac{4\sin 2\theta_\psi}{\gamma+1}\left(\gamma+1-2\gamma C_p - \frac{4}{M_\infty^2}\right) + \frac{1}{\gamma M_\infty^2}\left(\frac{d\frac{p}{p_\infty}}{d\theta}\right)_{\theta=\theta_\psi}\times}{\frac{8C_p}{\gamma+1}\left(1+\gamma\sin^2\theta_\psi - \frac{1}{M_\infty^2}\right) -}\rightarrow\right.$$

$$\left.\rightarrow\frac{\times\left[6C_p - \frac{8}{\gamma+1}\left(1+\gamma\sin^2\theta_\psi - \frac{1}{M_\infty^2}\right)\right]}{-3C_p^2 - 4\left(1-\frac{4}{\gamma+1}\,\frac{1}{M_\infty^2}\right)\sin^2\theta_\psi}\right\}, \tag{15}$$

where $C_p = \frac{2(\bar{p}_1 + h_1 - 1)}{\gamma M_\infty^2}$ and $\theta_\psi = \theta + \operatorname{arctg}\frac{\psi}{\bar{p}_1 + h_1}$. From [5] for laminar flow in the boundary layer on the dividing streamline and without heat transfer from the surface of the imaginary wedge with semivertex angle θ_ψ, we obtain

$$\frac{d\delta^*}{dx} = d_{\text{first}}\sqrt{C_{\text{first}}}\,\frac{M_\infty^2}{\sqrt{\text{Re}_{x_{\text{first}}}}} = d_{\text{first}}\frac{\bar{\chi}_{\text{first}}}{M_{\text{first}}}, \tag{16}$$

where

$$d_{\text{first}} = \frac{A(\text{Pr})}{M_{\text{first}}^2}\,\frac{T_{\theta_\psi}}{T_{\text{first}}} + (\gamma-1)B(\text{Pr}),$$

$$\frac{T_{\theta_\psi}}{T_{\text{first}}} \approx 1 + \sqrt{\text{Pr}}\,\frac{\gamma-1}{2}\,M_{\text{first}}^2, \tag{17}$$

$$C_{\text{first}} = \left(\frac{\mu_{\theta_\psi}}{\mu_{\text{first}}}\right)\left(\frac{T_{\text{first}}}{T_{\theta_\psi}}\right).$$

Here μ is determined from the Sutherland formula, and A(Pr) and B(Pr) are functions of Prandtl number and are given in [5] for Pr = 1 and 0.725. The induced pressure, calculated from Eq. (13) for the appropriate value of x, will be the second approximation. From it we find the increment in the angle caused by the blowing of gas, using the relation

$$\Delta\theta_2 = \operatorname{arctg}\frac{\psi}{\frac{p}{p_\infty}}.$$

This approximation corresponds to the imaginary wedge surface with semivertex angle $\theta + \Delta\theta_2$.

A third approximation for the pressure is obtained for the same value of x using Eq. (13). Here the flow parameters without viscous effects in Eqs. (13)-(17) are determined using the angle $\theta_\psi = \theta + \Delta\theta_2$. The formula for calculating the pressure reduces to the form

$$\frac{p_{\text{first}}}{p_\infty} = 1 + \frac{\gamma M_\infty^2}{2}(C_{p_2} + h),$$

where

$$C_{p_2} = \frac{1}{2}\left(\frac{A}{\beta} - C_{p_*}\right) - \sqrt{\frac{1}{4}\left(\frac{A}{\beta} - C_{p_*}\right)^2 - C_{p_*}\left(C_{p_*} - \frac{A}{\beta}\right) - \frac{B}{\beta^2}};$$

$$C_{p_*} = \frac{B - 0.5625 - \sqrt{(B - 0.5625)^2 + 4 \cdot 0.9687\,(A + 1.5)}}{2\beta(A + 1.5)};$$

$$\beta = \sqrt[3]{\frac{(\gamma + 1) M_\infty^2}{16 \sin^2 \theta_\psi}};\; A = \frac{4\beta}{\gamma + 1}\left(1 + \gamma \sin^2 \theta_\psi - \frac{1}{M_\infty^2}\right);$$

$$B = 4\beta^2 \sin^2 \theta_\psi \left(1 - \frac{4}{\gamma + 1}\frac{1}{M_\infty^2}\right);\; \frac{1}{h} = -\frac{f'(C_{p_2})}{f(C_{p_2})} + \frac{1}{2}\frac{f''(C_{p_2})}{f'(C_{p_2})};$$

$$f(C_p) = C_p^3 - \frac{A}{\beta}C_p^2 + \frac{B}{\beta^2}C_p + \frac{1}{\beta^3}.$$

The next approximations are calculated similarly. Clearly, the corresponding approximations for the pressures on the dividing streamline form two intersecting series, tending to the same limit. The odd approximations tend to the limit from the left and the even approximations, from the right.

3. Third Approximation (Strong Interaction)

Strong interaction for bodies of general shape is characterized by quite large Mach number and quite low values of Re_x. In our case the static pressure p/p_∞ and the displacement thickness δ^*/x along the outer edge of the boundary layer are given by the asymptotic series

$$\frac{p}{p_\infty} = p_0 \bar{\chi}\left[1 + \frac{p_1 K_{\theta_\psi}}{\bar{\chi}^{\frac{1}{2}}} + \frac{p_2 + p_3 K_{\theta_\psi}^2}{\bar{\chi}} + O\left(\bar{\chi}^{-\frac{3}{2}}\right)\right], \tag{18}$$

$$\frac{\delta^*}{x} = \delta_0 \frac{\bar{\chi}^{\frac{1}{2}}}{M_\infty}\left[1 + \frac{\delta_1 K_{\theta_\psi}}{\bar{\chi}^{\frac{1}{2}}} + \frac{\delta_2 + \delta_3 K_{\theta_\psi}^2}{\bar{\chi}} + O\left(\bar{\chi}^{-\frac{3}{2}}\right)\right], \tag{19}$$

where $K_{\theta_\psi} = M_\infty\left(\theta + \operatorname{arc\,tg}\frac{\psi}{\bar{p}_1 + h_1}\right)$; $\bar{\chi} = \frac{M_\infty^2 \sqrt{C_\infty}}{\sqrt{\mathrm{Re}_{x_\infty}}}$; and $p_0, p_1, p_2, p_3, \delta_0, \delta_1, \delta_2, \delta_3$ are unknown constants, determined by solving the boundary-layer equations, and in the tangent wedge approximation are linked by the relations

$$p_0 = \frac{9}{32}\gamma(\gamma + 1)\delta_0^2,\; p_1 = \frac{8}{3}\left(\delta_1 + \frac{1}{\delta_0}\right),\; p_2 = \frac{10}{3}\delta_2 + $$

$$+ \frac{32}{9}\frac{3\gamma + 1}{\gamma(\gamma + 1)^2}\frac{1}{\delta_0^2},\; p_3 = \frac{10}{3}\delta_3 + \frac{16}{9}\left(\delta_1 + \frac{1}{\delta_0}\right)^2. \tag{20}$$

Values of the latter constants have been calculated and tabulated, for example, in [6] for $\gamma = 1.4$ and values of Pr of 1 and 0.72, for different thermal conditions on the wedge wall. Therefore, the second approximation for the pressure in our case is found from Eq. (18) for the appropriate value of x, with the given value of $\frac{T_{0+\Delta 0_1}}{T_0}$, and from the pressure we calculate the angular increment

$$\Delta\theta_2 = \operatorname{arctg}\frac{\psi}{\frac{p}{p_\infty}} \text{ and } \theta_\psi = \theta + \Delta\theta_2.$$

From these data, for the same value of x, in complete analogy with the weak interaction case, we determine a third approximation for p/p_∞ from the determinant parameter $K_{\theta_\psi} = M_\infty(\theta + \Delta\theta_2)$. Thus, the next approximations are determined.

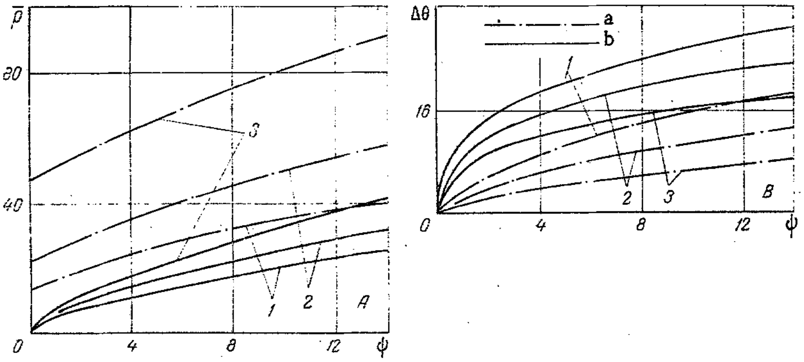

Fig. 1. The dimensionless pressure $\bar{p}$ on the dividing streamline (a) and the increment of angle $\Delta\theta$ (b) as a function of the blowing parameter ψ, for different positions of the wall slope and unperturbed flow Mach numbers: a) $\theta = 20°$, b) $0°$; 1) $M_\infty = 7$; 2) 10; 3) 15.

TABLE 1. Values of Dimensionless Pressure at the Shock and Effective Wedge Angle as a Function of Approximation Number

Approximation No.	$\frac{p}{p_\infty}$	$\Delta\theta$, deg	$\theta+\Delta\theta$, deg
1	19,04	16,66	26,66
2	23,00	13,92	23,92
3	20,30	15,61	25,61
4	21,90	14,59	24,59
5	21,05	15,17	25,17
6	21,35	14,95	24,95

For comparison, Fig. 1 shows part of the computations performed, in the form of $\bar{p}$ and $\Delta\theta$ as a function of different positions of the wedge wall, and for different blowing parameters and oncoming stream values for the case $Re_x \to \infty$ (first approximation). When the viscous interaction is allowed for, 5-6 approximations were enough to compute a single position of $\Delta\theta$. By way of example, Table 1 shows one of these calculations, illustrating the speed of convergence of the method under the following conditions: $\theta = 10°$; $M_\infty = 7$; $Pr = 0.725$; $\psi = 5.7$; $x = 0.005$ m; and $H = 30$ km.

NOTATION

M_∞, Mach number of unperturbed stream; γ, adiabatic index; $c_p = \frac{2(\bar{p}-1)}{\gamma M_\infty^2}$; $\bar{p} = \frac{p}{p_\infty}$; p, pressure; m, mass flow rate of blown gas, g/cm^2·sec; T_w, temperature of blowing gas; T_0, stagnation temperature; $B = 0.468$; $R_0 = 8.314 \cdot 10^3$ J/deg·k·mole (gas constant); M, molecular weight of blown gas; $\theta_\psi = \theta + \Delta\theta = \theta_w \pm \alpha + \Delta\theta$; θ_w, wedge semivertex angle; α, angle of attack; $\Delta\theta$, increment of angle due to blowing; $K\theta_\psi = M_\infty \theta_\psi$; $K_\theta = M_\infty \theta$; $K_\psi = M_\infty \psi$; Re_x, local Reynolds number; Pr, Prandtl number. Indices: θ_ψ, ∞, conditions on the contact surface and in the undisturbed flow.

LITERATURE CITED

1. J. A. Bott, AIAA J. 6, No. 4 (1968).
2. R. A. Hartunian and D. J. Spencer, AIAA J., 5, No. 8 (1967).
3. E. Karafoli, High-Speed Aerodynamics [in Russian], Izd. Akad. Nauk SSSR, Moscow (1960).
4. K. Lantsosh, Practical Methods of Applied Analysis [in Russian], Fizmatgiz, Moscow (1961).
5. W. D. Hayes and R. F. Probstein (editors), Hypersonic Flow Theory, Academic Press (1967).
6. T. K. Chattopadhyay and C. M. Rodkiewicz, AIAA J., 9, No. 3 (1971).